MECHANICAL ENGINEERING (V23-IInd YEAR COURSE STRUCTURE & SYLLABUS)

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3	0	0	3

II Year I Semester

NUMERICAL METHODS AND TRANSFORM TECHNIQUES

Course Objectives:

- To elucidate the different numerical methods to solve nonlinear algebraic equations
- To disseminate the use of different numerical techniques for carrying out numerical integration.
- To furnish the learners with basic concepts and techniques at plus two level to lead them into advanced level by handling various real world applications.

Course Outcomes:

1. Evaluate the approximate roots of polynomial and transcendental equations by different algorithms. Apply Newton's forward & backward interpolation and Lagrange's formulae for equal and unequal intervals (L3)
2. Apply numerical integral techniques to different Engineering problems. Apply different algorithms for approximating the solutions of ordinary differential equations with initial conditions to its analytical computations (L3)
3. Apply the Laplace transform for solving differential equations (L3)
4. Find or compute the Fourier series of periodic signals (L3)
5. Know and be able to apply integral expressions for the forwards and inverse Fourier transform to a range of non-periodic waveforms (L3)

UNIT – I: Iterative Methods:
Introduction – Solutions of algebraic and transcendental equations: Bisection method – Secant method – Method of false position – Iteration method – Newton-Raphson method (One variable &Simultaneous Equations)
Interpolation: Newton's forward and backward formulae for interpolation –Central differences-Bassels& Stirling formulas- Interpolation with unequal intervals – Lagrange's interpolation formula

UNIT – II: Numerical integration, Solution of ordinary differential equations with initial conditions:
Trapezoidal rule– Simpson's 1/3rd and 3/8th rule– Solution of initial value problems by Taylor's series– Picard's method of successive approximations
Single step method- Euler's method –Modified Euler's method-Runge- Kutta method (second and fourth order)
Multi step method –Milne's Predictor and Corrector Method.

UNIT –III: Laplace Transforms:

Definition of Laplace transform - Laplace transforms of standard functions – Properties of Laplace Transforms – Shifting theorems–Transforms of derivatives and integrals – Unit step function – Dirac's delta function – Inverse Laplace transforms – Convolution theorem (with out proof).

Applications: Solving ordinary differential equations (initial value problems& boundary value problems) and integro differential equations using Laplace transforms.

UNIT – IV: Fourier series:

Introduction– Periodic functions – Fourier series of periodic function –Dirichlet's conditions – Even and odd functions –Change of interval– Half-range sine and cosine series.

UNIT – V: Fourier Transforms:

Fourier integral theorem (without proof) – Fourier sine and cosine integrals – Infinite Fourier transforms – Sine and cosine transforms – Properties– Inverse transforms – Convolution theorem (without proof) – Finite Fourier Cosine& sine transforms.

Text Books:

1. **B. S. Grewal,** Higher Engineering Mathematics, 44th Edition, Khanna Publishers.
2. **B. V. Ramana,**Higher Engineering Mathematics, 2007 Edition, Tata Mc. Graw Hill Education.

Reference Books:

1. **Erwin Kreyszig,**Advanced Engineering Mathematics, 10th Edition, Wiley-India.
2. **Steven C. Chapra,** Applied Numerical Methods with MATLAB for Engineering and Science,Tata Mc. Graw Hill Education.
3. **M. K. Jain, S.R.K. Iyengar and R.K. Jain,** Numerical Methods for Scientific and Engineering Computation, New Age International Publications.
4. **Lawrence Turyn,**Advanced Engineering Mathematics, CRC Press.

Unit-II Simpson's Rule

$\frac{1}{3}$rd

Numerical Integraton & Solution of oridnary differential equations

Trapizoidal Rule:

Let $f(x)$ be the function and the points (x_0, y_0) (x_1, y_1) (x_2, y_2) .... (x_n, y_n) are in the value $y = f(x)$ then.

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots\cdots+y_{n-1})\right]$$

Where $h = \frac{b-a}{n}$.

~~Wher~~ $h = \frac{x_n - x_0}{n} = \frac{b-a}{n}$

this known as Trapizoidal Rule formula

① Evalute $\int_0^1 x^3\,dx$ with 5 sub internals by using trapizoidal Rule.

Sol: Given that $\int_0^1 x^3\,dx$

$f(x) = x^3$ $\qquad x_0 = 0$

$x_1 = x_0 + h$

$y = f(x)$

$a = 0, \; b = 1, \; x_0 = 0$

$y_0 = f(x_0) = x_0^3 = (0)^3 = 0.$

5 Subintervals $n=5$

By trapezoidal rule

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots\cdots y_{n-1})\right].$$

$h=0.2$, $x_0=0$.

$$h = \frac{b-a}{n} = \frac{1-0}{5} = \frac{1}{5}$$
$$h = 0.2$$

$y_0 = f(x_0) = 0$

$x_1 = x_0+h = 0+0.2 = 0.2$

$x_2 = x_1+h = 0.2+0.2 = 0.4$

$x_3 = x_2+h = 0.4+0.2 = 0.6$

$x_4 = x_3+h = 0.6+0.2 = 0.8$

$x_5 = x_4+h = 0.8+0.2 = 1$

$f(x) = x^3$

$y_0 = f(x_0) = (x_0)^3 = (0)^3 = 0$

$y_0 = f(x_0) = (x_1)^3 = (0.2)^3 = 0.008$

$y_1 = f(x_1) = (x_2)^3 = (0.4)^3 = 0.064$

$y_2 = f(x_2) = (x_3)^3 = (0.6)^3 = 0.216$

$y_3 = f(x_3) = (x_4)^3 = (0.8)^3 = 0.512$

$y_4 = f(x_4) = \quad 1$

x' and 'y' values in the table

	x_0	x_1	x_2	x_3	x_4	x_5
x	0	0.2	0.4	0.6	0.8	1
y	0	0.008	0.064	0.216	0.512	1
	y_0	y_1	y_2	y_3	y_4	y_5

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\dots y_{n-1})\right]$$

$$\int_0^1 x^3\,dx = \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{0.2}{2}\left[(0+1)+2(0.008+0.064+0.216+0.512)\right]$$

$$= 0.1\,[1+1.6]$$

$$= 0.26$$

Verification:

$$\int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1$$

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1}}$$

$$= \left[\frac{(1)^4}{4} - \frac{(0)^4}{4}\right]$$

$$= \frac{1}{4} - 0$$

$$= 0.25$$

By Simpson's $1/3^{rd}$ rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+y_6 \cdots) + 4(y_1 + y_3+y_5+\cdots)\right]$$

Problem

① $\int_0^{\pi} \sin x\,dx$ by using Simpson's $1/3^{rd}$ rule 10 equal parts.

Sol:
$$f(x) = \sin x.$$

$$x_0 = a = 0, \quad b = \pi, \quad n = 10$$

$$y_0 = f(x_0) = \sin x_0 = \sin 0 = 0$$

$$h = \frac{b-a}{n} = \frac{\pi - 0}{10} = \frac{\pi}{10} = 0.314$$

$$x_1 = x_0 + h = 0 + 0.314 = 0.314$$

$$x_2 = x_1 + h = 0.314 + 0.314 = 0.628$$

$$x_3 = x_2 + h = 0.628 + 0.314 = 0.942$$

$$x_4 = x_3 + h = 0.942 + 0.314 = 1.256$$

$$x_5 = x_4 + h = 1.256 + 0.314 = 1.57$$

$$x_6 = x_5 + h = 1.57 + 0.314 = 1.884$$

$$x_7 = x_6 + h = 1.884 + 0.314 = 2.198$$

$x_8 = x_7 + h = 2.198 + 0.314 = 2.512$

$x_9 = x_8 + h = 2.512 + 0.314 = 2.826$

$x_{10} = x_9 + h = 2.826 + 0.314 = 3.14.$

$y_0 = f(x_0) = \sin x = \sin 0 = 0$

$y_1 = f(x_1) = \sin x_1 = \sin 0.314 = 0.308$

$y_2 = f(x_2) = \sin x_2 = \sin 0.628 = 0.587$

$y_3 = f(x_3) = \sin x_3 = \sin 0.942 = 0.808$

$y_4 = f(x_4) = \sin x_4 = \sin 1.256 = 0.950$

$y_5 = f(x_5) = \sin x_5 = \sin 1.57 = 0.999$

$y_6 = f(x_6) = \sin x_6 = \sin 1.884 = 0.951$

$y_7 = f(x_7) = \sin x_7 = \sin 2.198 = 0.809$

$y_8 = f(x_8) = \sin x_8 = \sin 2.512 = 0.588$

$y_9 = f(x_9) = \sin x_9 = \sin 2.826 = 0.310$

$y_{10} = f(x_{10}) = \sin x_{10} = \sin 3.14 = 0.001$

$$\int_0^{\pi} \sin x \cdot dx = \frac{h}{3}\left[(y_0+y_{10})+2(y_2+y_4+y_6+y_8)+4(y_1+y_3+y_5+y_7+y_9)\right]$$

$$= \frac{0.314}{3}\left[(0+0.001)+2(0.587+0.950+0.951+0.588)+4(0.308+0.808+0.999+0.809+0.361)\right]$$

$$= 1.985$$

Verification

$$\int_0^{\pi} \sin x \cdot dx = \left[-\cos x\right]_0^{\pi}$$

$$= \left[-\cos\pi + \cos 0\right]$$

$$= \left[-(-1)+1\right]$$

$$= 2$$

Simpson's 3/8 Rule :-

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\left[(y_0+y_n)+2(y_3+y_6+y_9+\dots y_n)+3(y_1+y_2+y_4+y_5+\dots y_{n-1})\right]$$

Problem:

① $\int_0^6 \frac{1}{1+x} dx$ with 6 Parts by Simpson's 3/8 rule.

Sol: Here

$$let \ f(x) = \frac{1}{1+x}$$

$$x_0 = a = 0, \quad b = 6, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{6-0}{6} = 1 \quad \boxed{h=1}$$

$$x_0 = 0.$$

$$x_1 = x_0 + h = 0 + 1 = 1$$

$$x_2 = x_1 + h = 1 + 1 = 2$$

$$x_3 = x_2 + h = 2 + 1 = 3$$

$$x_4 = x_3 + h = 3 + 1 = 4$$

$$x_5 = x_4 + h = 4 + 1 = 5$$

$$x_6 = x_5 + h = 5 + 1 = 6.$$

$$y_0 = f(x_0) = \frac{1}{1+x_0} = \frac{1}{1+0} = 1$$

$$y_1 = f(x_1) = \frac{1}{1+x_1} = \frac{1}{1+1} = 0.5$$

$$y_2 = f(x_2) = \frac{1}{1+x_2} = \frac{1}{1+2} \doteq 0.333$$

$$y_3 = f(x_3) = \frac{1}{1+x_3} = \frac{1}{1+3} = 0.25$$

$$y_4 = f(x_4) = \frac{1}{1+x_4} = \frac{1}{1+4} = 0.2$$

$$y_5 = f(x_5) = \frac{1}{1+x_5} = \frac{1}{1+5} = 0.166$$

$$y_6 = f(x_6) = \frac{1}{1+x_6} = \frac{1}{1+6} = 0.142$$

$$\cancel{\int_0^6 \frac{1}{1+x} = \frac{3}{8}\left[(y_0+y_6)+2(0.25)\ 0.142 + 3(0.5\right.}$$

$$\int_0^6 \frac{1}{1+x}\,dx = \frac{3h}{8}\left[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3\times1}{8}\left[(1+0.142)+2(0.25)+3(0.5+0.333+0.2+0.166)\right]$$

$$= 1.964.$$

Numerical Solution for the ordinary differential Equations Picard's method.

Consider $\frac{dy}{dx} = f(x, y)$ then

$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$ is called first approximation

$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$ is called second approximation

$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\, dx$ is called third approximation

$y^{(4)} = y_0 + \int_{x_0}^{x} f(x, y^{(3)})\, dx$ is called fourth "

(1) Using Picard's method to find the value of y and $x=0$, $x=0.2$ given $\frac{dy}{dx} = x-y$ if initial condition $y=1$ when $x=0$

Sol:-

Given :-

$\frac{dy}{dx} = x-y$ if initial condition

$y=1$ when $x=0$

$$f(x,y) = x-y \longrightarrow (1)$$

Given $y=1$ when $x=0$

$\Rightarrow x_0 = 0, \quad y_0 = 1$

By Picard's Method.

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx.$$

$$f(x, y_0) = x - y_0$$

$$f(x, y_0) = x - 1 \qquad (x_0 = 0)$$

$$y^{(1)} = 1 + \int_0^x (x-1)\, dx$$

$$y^{(1)} = 1 + \int_0^x \left[\frac{x^2}{2} - x\right]$$

$$y^{(1)} = 1 + \left[\frac{x^2}{2} - x\right]_0^x$$

$\int x \cdot dx = \frac{x^2}{2}$

$\int x^n dx = \frac{x^{n+1}}{n+1}$

$\int 1 \cdot dx = x$

$$y^{(1)} = 1 + \left[\frac{x^2}{2} - x\right]$$

$$\boxed{y^{(1)} = 1 - x + \frac{x^2}{2}} \quad (2)$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx.$$

$f(x, y) = x - y$

$$f(x, y^{(1)}) = x - y^{(1)}$$

$$= x - \left[1 - x + \frac{x^2}{2}\right]$$

$$= x - 1 + x - \frac{x^2}{2}$$

$$f(x, y^{(1)}) = 2x - 1 - \frac{x^2}{2}$$

$$y^{(2)} = 1 + \int_0^x \left(2x - 1 - \frac{x^2}{2}\right) dx$$

$\int 2x = \int x^1 \, dx = 2\frac{x^2}{2} = x^2$

$\int 1 \, dx = x$

$$= 1 + \left[x^2 - x - \frac{x^3}{6}\right]_0^x$$

$$= 1 + \left[-x + x^2 - \frac{x^3}{6}\right]$$

$$\boxed{y^{(2)} = 1 - x + x^2 - \frac{x^3}{6}} \longrightarrow (3)$$

$$y^{(3)} = y_0 + \int_{x_0}^x f(x, y^{(2)}) \, dx$$

$$f(x, y^{(2)}) = x - y^{(2)}$$

$$= x - \left[1 - x + x^2 - \frac{x^3}{6}\right]$$

$$= x - 1 + x - x^2 + \frac{x^3}{6}$$

$$f(x, y^{(2)}) = 1 + 2x - x^2 + \frac{x^3}{6}$$

$$y^{(3)} = 1 + \int_0^x \left(1 + 2x - x^2 + \frac{x^3}{6}\right) dx$$

$\int \frac{x^3}{6} dx = \frac{1}{6}\int x^3 dx$

$\frac{1}{6} \cdot \frac{x^4}{4}$

$= \frac{x^4}{24}$

$$= 1 + \left[x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}\right]_0^x$$

$$= 1 + \left[x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}\right] - 0$$

$$y^{(3)} = 1 + x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}$$

$$y^{(4)} = y_0 + \int_{x_0}^{x} f(x, y^{(3)})\,dx \qquad f(x,y) = x - y$$

$$= f(x, y^{(3)}) = x - y^{(3)}$$

$$= x - \left[1 + x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}\right]$$

$$= x - 1 - x - x^2 + \frac{x^3}{3} - \frac{x^4}{24}$$

$$f(x, y^{(3)}) = 1 - x^2 + \frac{x^3}{3} - \frac{x^4}{24}$$

$$y^{(4)} = 1 + \int_0^x \left(1 - x^2 + \frac{x^3}{3} - \frac{x^4}{24}\right) dx$$

$$= 1 + \left[x - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}\right]_0^x$$

$$y^{(4)} = \left[1 + x - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}\right] - 0$$

$$\boxed{y^{(4)} = 1 + x - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}}$$

$$\int \frac{x^3}{3}\,dx = \frac{1}{3}\int x^3\,dx = \frac{1}{3}\cdot\frac{x^4}{4} = \frac{x^4}{12}$$

$$\int \frac{x^4}{24}\,dx = \frac{1}{24}\int x^4\,dx = \frac{1}{24}\cdot\frac{x^5}{5} = \frac{x^5}{120}$$

$$y = 1 + x - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}$$

② find the solution of $\frac{dy}{dx} = 1+xy$; $y(0)=1$ in the interval $[0, 0.05)$ correct to three decimal places ~~using Picard's method~~. taking $h=0.1$.

Sol:

Given $\frac{dy}{dx} = 1+xy \longrightarrow ①$

$y(0)=1$; $x_0=0$; $y_0=1$; $h=0.1$

By Picard's Method

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\, dx$$

$$f(x, y) = 1+xy$$
$$= 1+x\, y_0$$

Since $y_0=1$

$$f(x, y) = 1+x(1)$$
$$= 1+x$$

$$y^{(1)} = 1 + \int_0^x (1+x)\, dx$$

$$= 1\left[x + \frac{x^2}{2}\right]_0^x$$

$$= 1\left[x + \frac{x^2}{2}\right]$$

$$y^{(1)} = 1+x+\frac{x^2}{2} \longrightarrow ②$$

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$$

$$f(x, y^{(1)}) = 1+xy$$

$$= 1+xy^{(1)}$$

$$f(x, y^{(1)}) = 1+x\left(1+x+\frac{x^2}{2}\right)$$

$$= 1+x+x^2+\frac{x^3}{2}$$

$$y^{(2)} = 1+\int_{0}^{x}\left(1+x+x^2+\frac{x^3}{2}\right) dx$$

$$= 1\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}\right]_0^x$$

$$= 1\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}\right]-0$$

$$y^{(2)} = 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8} \longrightarrow 3$$

$$y^{(3)} = y_0+\int_{x_0}^{x} f(x, y^{(2)})\, dx$$

$$f(x, y^{(2)}) = 1+xy$$

$$= 1+xy^{(2)}$$

$$f(x, y^{(2)}) = 1+x\left(1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}\right)$$

$$f(x, y^{(2)}) = 1+x+x^2+\frac{x^3}{2}+\frac{x^4}{3}+\frac{x^5}{8}$$

$$y^{(3)} = 1+\int_{0}^{x}\left(1+x+x^2+\frac{x^3}{2}+\frac{x^4}{3}+\frac{x^5}{8}\right) dx.$$

$$= \left[x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{x^6}{48}\right]_0^x$$

$$y^{(3)} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

$$y = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

$$x_1 = x_0 + h$$
$$= 0 + 0.1$$
$$= 0.1$$

$$y(0.1) = 1 + (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + \frac{(0.1)^5}{15} + \frac{(0.1)^6}{48}$$

RUNGE-KUTTA Method for Second order differential equation:

Consider for Example the Second order diffe equation.

$$y'' = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0$$

Substituting $\frac{dy}{dx} = z$ —(1)

$$\frac{dz}{dx} = \frac{d^2y}{dx^2} = f(x, y, z) \text{ using (1)} \quad \text{—(2)}$$

Given $y(x_0) = y_0$ and $y'(x_0) = z(x_0) = y'_0$

Equation (1) & (2) Constitute the ~~equatio~~ equivalent System of Simultaneous equations

where $f_1(x, y, z) = z$, $f_2(x, y, z) = f(x, y, z)$

given also $y(0)$ and $z(0)$.

① Solve $y'' - x(y')^2 + y^2 = 0$ using Runge-Kutta (second order method) of $x = 0.2$. Given $y(0) = 1$, $y'(0) = 0$ taking $h = 0.2$

Sol: Given equation is a second order DE

Substituting $y' = \frac{dy}{dx} = z = f_1(x, y, z) \longrightarrow ①$

Given $y'' - x(y')^2 + y^2 = 0$

$$\frac{d^2y}{dx^2} - xz^2 + y^2 = 0$$

$$\frac{d^2y}{dx^2} = xz^2 - y^2 = f_2(x, y, z) \longrightarrow ②$$

Given $x_0 = 0$, $y_0 = 1$, $z_0 = y'(0) = 0$ and $h = 0.2$

By R.K algorithm

$$k_1 = hf_1(x_0, y_0, z_0)$$
$$= 0.2 f_1(0, 1, 0)$$
$$= 0.2 \times 0$$
$$\boxed{k_1 = 0}$$

$$l_1 = hf_2(x_0, y_0, z_0) = 0.2 \times f_2(0, 1, 0)$$
$$= 0.2 \times [0(0)^2 - (1)^2]$$
$$= 0.2(-1)$$
$$\cancel{-0.2} = -0.2$$

$k_2 = hf_1\left(x_0+\frac{1}{2}h,\ y_0+\frac{1}{2}k_1,\ z_0+\frac{1}{2}l_1\right)$

$= 0.2\, f_1\left(0+\frac{0.2}{2},\ 1+\frac{0}{2},\ 0+\frac{(-0.2)}{2}\right).$

$= 0.2\, f_1(0.1,\ 1,\ -0.1)$

$= 0.2 \times -0.1$

$\boxed{k_2 = -0.02}$

$l_2 = hf_2\left(x_0+\frac{1}{2}h,\ y_0+\frac{1}{2}k_1,\ z_0+\frac{1}{2}l_1\right)$

$= 0.2\, f_2(0.1,\ 1,\ -0.1)$

$= 0.2\left[(0.1)(-0.1)^2-1\right]$

$= 0.2\left[0.1\times 0.01-1\right]$

$= 0.2\ [-0.99]$

$\boxed{l_2 = -0.1998}$

$k_3 = hf_1\left(x_0+\frac{1}{2}h,\ y_0+\frac{1}{2}k_2,\ z_0+\frac{1}{2}l_2\right)$

$= 0.2\ f_1\left(0+\frac{0.2}{2},\ 1+\frac{-0.02}{2},\ 0+\frac{-0.1998}{2}\right).$

$= 0.2\ f_1(0.1,\ 0.99,\ -0.0999)$

$= 0.2 \times -0.0999$

$\boxed{k_3 = -0.01998.}$

$l_3 = hf_2\left(x_0+\frac{1}{2}h,\ y_0+\frac{1}{2}k_2,\ z_0+\frac{1}{2}l_2\right)$

$= 0.2\ f_2(0.1,\ 0.99,\ -0.0999)$

$= 0.2\times(-0.9791)$

$= -0.1958$

$k_4 = hf_1(x_0+h, y_0+k_3, z_0+l_3)$

$= 0.2\, f_1(0.2, 0.98, -0.1958)$

$= 0.2(-0.1958)$

$k_4 = -0.0392$

$l_4 = hf_2(x_0+h, y_0+k_3, z_0+l_3)$

$= 0.2\, f_2(0.2, 0.98, -0.1958)$

$= 0.2\,[(0.2)(-0.1958) - (0.98)]$

$l_4 = -0.1905$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y(0.2) = 1 + \frac{1}{6}\; 0 + 2(-0.02) + 2(0.1998) - 0.190$$

$= 0.98014.$

② Use R.K method to find $y(0.1)$ for the equation $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$

Rk (Rangi - kutta) method of 4^{th} order

Consider $\frac{dy}{dx} = f(x, y)$

$$y = y_0 + k$$

$$k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$k_1 = h.f(x_0, y_0)$$

$$k_2 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_3}{2}\right)$$

Similarly $\boxed{y_2 = y_1 + k}$

$$k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$k_1 = h.f(x_1, y_1)$$

$$k_2 = h.f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = h.f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = h.f(x_1 + h, y_1 + k_3)$$

* Problem:

use R.k method of 4th order find the value of y at x = 0.1, h = 0.1

given, $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$

Sol Given data,

$$\frac{dy}{dx} = \frac{y-x}{y+x}$$

$$f(x,y) = \frac{y-x}{y+x}$$

$x_0 = 0$, $y_0 = 1$ & $h = 0.1$

$y(0) = 1$

$k_1 = h.f(x_0, y_0)$

$= 0.1 \times \frac{1-0}{1+0}$

$= 0.1 \times 1$

$$\boxed{k_1 = 0.1}$$

$$k_2 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \quad \text{Formula}$$

$$= 0.1 \times F\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$

$$= 0.1 \times F\left(\frac{0.1}{2}, \frac{2.1}{2}\right)$$

$$= 0.1 \times F(0.05, 1.05)$$

$$= 0.1 \times \frac{1.05 - 0.05}{1.05 + 0.05}$$

$$= 0.1 \times \frac{1}{1.1}$$

$$\boxed{k_2 = 0.0909}$$

$$k_3 = h.F\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= h.f\left(0 + \frac{0.1}{2}, 1 + \frac{0.0909}{2}\right)$$

$$= h.F\left(\frac{0.1}{2}, \frac{2.0909}{2}\right)$$

$$= 0.1 \times \frac{1.045545 - 0.05}{1.04545 + 0.05}$$

$$= 0.1 \times 0.99545$$

$$\boxed{k_3 = 0.0909}$$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= hf(0+0.1, 1+0.0909)$

$= hf(0.1, 1.0909)$

$= 0.1 \times \frac{1.0909 - 0.1}{1.0909 + 0.1}$

$= 0.1 \times \frac{0.9909}{1.1909}$

$\boxed{k_4 = 0.0832}$

$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}[0.1 + 2(0.909) + 2(0.909) + 0.0832]$

$k = \frac{0.5468}{6}$

$k = 0.9113$

$\boxed{\therefore k = 0.0911}$

$y_1 = y_0 + k$

$y_1 = 1 + 0.0911$

$y_1 = 1.0911$

$x_1 = 0.1$

Taylor's Series method :-

To find the numerical solution of the diff. equation

$$\frac{dy}{dx} = f(x, y) \longrightarrow (1)$$

Given the initial condition $y(x_0) = y_0$

$$y(x) = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots$$

$$y_1 = y_0 + \frac{h}{h!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

Similarly

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots$$

(1) Using Taylor's series method, solve the equation $\frac{dy}{dx} = x^2 + y^2$ for $x = 0.4$ given that $y = 0$ when $x = 0$

Sol: $y' = f(x, y)$

Given $f(x, y) = x^2 + y^2$

$$y' = \frac{dy}{dx} = x^2 + y^2$$

$$y'' = \frac{d^2y}{dx^2} = 2x + 2yy'$$

$$y''' = 2(1) + 2[y.y'' + y'.y']$$

$$\Rightarrow y''' = 2 + 2[yy'' + (y')^2]$$

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(x^2) = 2x^{2-1} \Rightarrow 2x$

$\frac{d}{dx}(y^2) = 2y.y'$

$\frac{d}{dx}(x) = 1$

$\frac{d}{dx}(uv) = uv' + vu'$

The Taylor's series

$$y(x) = y(0) + \frac{h}{1!}y'(0) + \frac{h^2}{2!}y''(0) + \frac{h^3}{3!}y'''(0) + \cdots$$

$$= 0 + 0 + 0 + \frac{h^3}{3\times2\times1}\times 2 + \cdots$$

$y(x) = \frac{h^3}{\cancel{6}3}x2 +$ (higher power are negleted)

$$y(x) = \frac{h^3}{3}$$

$$y(0.4) = \frac{(0.4)^3}{3} = 0.0213$$

(2) Using the Taylor's series method solve

$\frac{dy}{dx} = 2y + 3e^x$; $y(0) = 0$ at

$x = 0.1, 0.2$

Sol: Given $y' = \frac{dy}{dx} + 3e^x$

$$y_0' = 2(0) + 3e^0 = 0 + 3\cdot1 = 3$$

$y'' = 2y' + 3e^x \Rightarrow y_0'' = 2(3) + 3e^0$

$y_0'' = 6 + 3 \times 1$

$\boxed{y_0'' = 9}$

$y''' = 2y'' + 3e^x$

$\Rightarrow y_0''' = 2(9) + 3e^0 = 18 + 3 \times 1$

$\boxed{y_0''' = 21}$

$y^{IV} = 2y''' + 3e^x$

$y_0^{IV} = 2(21) + 3 \times e^0$

$y_0^{IV} = 42 + 3 \times 1$

$\boxed{y_0^{IV} = 45}$

We have the Taylor's series algorithm,

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{IV} + \cdots$$

$$y(0.2) = 0 + \frac{0.2}{1} \times 3 + \frac{(0.2)^2}{2 \times 1} \times 9 + \frac{(0.2)^3}{3 \times 2 \times 1} \times 21 + \frac{(0.2)^4}{4 \times 3 \times 2 \times 1} \times 45 + \cdots$$

$y(0.2) = 0.6 + 0.18 + 0.028 + 0.003$

$= 0.811$

Euler's method:

① using Euler's method solve at $x=2$ from $\frac{dy}{dx} = 3x^2+1$; $y(1)=2$ making step size

(i) $h=0.5$ (ii) $h=0.25$

Sol: Here given $\frac{dy}{dx} = 3x^2+1$

let $f(x,y) = 3x^2+1$

Euler's algorithm is

$y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow ①$

(i) $h=0.5$

taking $n=0$ in (1) we have

$y_1 = y_0 + hf(x_0, y_0)$

Here $y_1 = 2$, $y_0 = 1$, $x_0 = 1$

$h = 0.5$

$x_1 = x_0 + h$

$x_1 = 1 + 0.5$

$x_1 = 1.5$

$y_1 = 2 + 0.5\, f(1,2)$

$y_1 = 2 + 0.5[3(1)^2+1]$

$y_1 = 2 + 0.5 \times 4$

$y_1 = 2+2$

$\boxed{y_1 = 4}$

Taking $n=1$ in Eq(1) we have

$y_2 = y_1 + h f(x_1, y_1)$

Here $y_1 = 4$ | $x_1 = x_0 + h$

$h = 0.5$ | $x_1 = 1 + 0.5$

$y_2 = 4 + 0.5 [3(1.5)^2 + 1]$ | $x_1 = 1.5$

$y_2 = 4 + 0.5 \times 7.75$

$\boxed{y_2 = 7.875}$

Taking $n=2$ in Eq(1) we have

$y_3 = y_2 + h f(x_2, y_2)$

Here $y_2 = 7.875$

$x_2 = x_1 + h$

$= 1.5 + 0.5$

$\boxed{x = 2}$

$h = 0.5$

$y_3 = 7.875 + 0.5 \times [3(2)^2 + 1]$

$= 7.875 + 0.5 \times 13$

$\boxed{y_3 = 14.375}$

ii) $h = 0.25$

Taking $n=0$ in (1) we have

$$y_1 = y_0 + h f(x_0, y_0)$$

$y_0 = 2$, $x_0 = 1$, $h = 0.25$

$$y_1 = 2 + 0.25 [3(1)^2 + 1]$$

$$y_1 = 2 + 0.25 \times 4$$

$$y_1 = 2 + 1$$

$$\boxed{y_1 = 3}$$

Taking $n=1$ in (1) we have

$$y_2 = y_1 + h \cdot f(x_1, y_1)$$

$y_1 = 3$, $h = 0.25$, $x_1 = x_0 + h$

$x_1 = 1 + 0.25$

$x_1 = 1.25$

$$y_2 = 3 + 0.25 \cdot [3(1.25)^2 + 1]$$

$$\boxed{y_2 = 4.425}$$

Taking $n=2$ in (1) we have

$$y_3 = y_2 + h f(x_2, y_2)$$

$y_2 = 4.425$, $h = 0.25$; $x_2 = x_1 + h$

$$y_3 = 4.425 + 0.25 [3(1.5)^2 + 1]$$

$= 1.25 + 0.25 = 1.5$

$$\boxed{y_3 = 6.357}$$

Taking $n=3$ in Eq ① we have

$y_4 = y_3 + h f(x_3, y_3)$

$y_3 = 6.357$; $h = 0.25$

$x_3 = x_2 + h$

$= 1.5 + 0.25$

$= 1.75$

$y_4 = 6.357 + 0.25 [3(1.75)^2 + 1]$

$\boxed{y_4 = 8.903}$

Conclusion:- The value of y at h=0.5 & h=0.25 the difference of operations are nearly 2.

HW

② Solve numerically using Euler's method $y' = y^2 + x$, $y(0) = 1$, find $y(0.1)$ and $y(0.2)$

Modify Euler's method :-

① Given $\frac{dy}{dx} = -xy^2$, $y(0) = 2$ compute $y(0.2)$ in step of 0.1 using modify Euler's method.

Sol:- Here $\frac{dy}{dx} = f(x,y) = -xy^2$

$x_0 = 0$, $y_0 = 2$ and $h = 0.1$

To find y_1 i.e, $y_1(0.1)$

$f(x_0, y_0) = -(0)(2)^2 = 0$

$y_1^{(0)} = y_0 + h\, f(x_0, y_0)$

$= 2 + 0.1 \times 0$

$\boxed{y_1^{(0)} = 2}$

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$; $f(x_0, y_0) = 0$

$f(x_1, y_1^{(0)}) = f(0.1, 2)$

$= -(0.1)(2)^2$

$= -0.1 \times 4$

$= -0.4$

$y_1^{(1)} = 2 + \frac{0.1}{2} \cdot [0 - 0.4]$

$\boxed{y_1^{(1)} = 1.98}$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

~~Yo~~ $y_0 = 2$, $h = 0.1$, $f(x_0, y_0) = 0$

$f(x_1, y_1^{(1)}) = f(0.1, 1.98) = -(0.1)(1.98)^2$

$= -0.39$

$= 2 + \frac{0.1}{2}[0 - 0.39]$

$= 2 + 0.05 \times \cancel{0} - 0.39$

$\boxed{y_1^{(2)} = 1.98}$

$y_1^{(1)} = y_1^{(2)} = 1.98$

$y(0.1) = 1.98$

To find y_2 i.e. $y(0.2)$

~~$x_1 = 0.1$, $y_1 = 1$~~ $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$h = 0.2$

$f(x_1, y_1) = f(0.2, 1.98) = -(0.2)(1.98)^2$

$= -0.784$

$Y_2 =$ i.e $y(0.2)$

$f(x_1, y_1) = f(0.2, 1.98)$

$= -0.2 (1.98)^2$

$\boxed{Y_2 = -0.784}$

MILNE predictor & corrector method:

$$Y_{n+1}, P = Y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

$$Y_{n+1}, C = Y_{n-2} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

problem:

① Given, that $\frac{dy}{dx} = 1+y^2$; $y(0) = 0$, $y(0.2) = 0.2027$, $y(0.4) = 0.4228$, $y(0.6) = 0.6841$ find $y(0.8)$ using milne predictor and corrector method.

Soln: Given $y' = \frac{dy}{dx} = 1+y^2$

$x_0 = 0$, $y_0 = 0$

$x_1 = 0.2$, $y_1 = 0.2027$

$x_2 = 0.4$, $y_2 = 0.4228$

$x_3 = 0.6$, $y_3 = 0.6841$

$$\boxed{y' = 1+y^2}$$

$y_0' = 1+(y_0)^2 = 1+(0)^2 = 1+0 = 1$

$y_1' = 1+(y_1)^2 = 1+(0.2027)^2 = 1.04108.$

$y_2' = 1+(y_2)^2 = 1+(0.4428)^2 = 1.1959$

$y_3' = 1+(y_3)^2 = 1+(0.6841)^2 = 1.4679$

$y_{n+1}, p = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n]$

Put $n=3$, $h=0.8$ to find $y(0.8)$

$y_{3+1}, p = y_{3-3} + \frac{4 \times 0.8}{3}[2y'_{3-2} - y'_{3-1} + 2y'_3)$

$y_4(0.8), p = y_0 + 1.0666[2y_1' - y_2' + 2y_3']$

$= 0+1.0666[2(1.04108) - 1.1959 + 2(1.4679)]$

$y(0.8) = 1.0666 \cdot (3.8220)$

$y(0.8) = 4.0766$

$y_{n+1}, c = y_{n-2} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$

put $n=3$, $h=0.8$ to find $y(0.8), c$

$$Y_{3+1,c} = Y_{3-2} + \frac{0.8}{3}[Y'_{3-1} + 4Y'_3 + Y'_{3+1}]$$

$$Y(0.8)_{,c} = Y_1 + 0.2666[Y_2' + 4Y_3' + Y_4']$$

$$Y(0.8)_{,c} = 0.2027 + 0.2666[1.1959 + 4(1.4679) + 17.6186]$$

$$\boxed{Y_4' = 1 + (Y_4)^2 = 1 + (4.0066)^2 = 17.6186}$$

$$Y(0.8)_{,c} = 6.784$$